Exercise 3

In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

$$(a)\ (-1)^{1/3}; \qquad (b)\ 8^{1/6}.$$

$$Ans.\ (b)\ \pm\sqrt{2},\ \pm\frac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\frac{1-\sqrt{3}i}{\sqrt{2}}.$$

Solution

Part (a)

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its third roots are

$$z^{1/3} = \left[re^{i(\Theta+2\pi k)}\right]^{1/3} = r^{1/3}\exp\left(i\frac{\Theta+2\pi k}{3}\right), \quad k = 0, 1, 2.$$

The magnitude and principal argument of -1 are respectively $r = 1$ and $\Theta = \pi$.

$$(-1)^{1/3} = 1^{1/3}\exp\left(i\frac{\pi+2\pi k}{3}\right), \quad k = 0, 1, 2$$

The first, or principal, root ($k = 0$) is

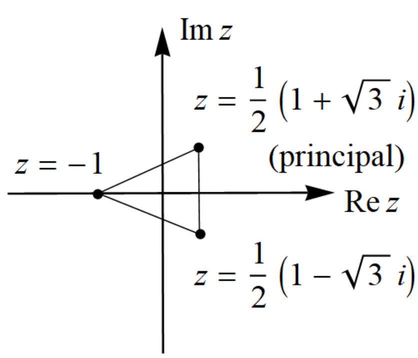
$$(-1)^{1/3} = 1^{1/3}e^{i\pi/3} = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1}{2}(1+\sqrt{3}i),$$

the second root ($k = 1$) is

$$(-1)^{1/3} = 1^{1/3}e^{i\pi} = 1(\cos\pi + i\sin\pi) = -1 + i(0) = -1,$$

and the third root ($k = 2$) is

$$(-1)^{1/3} = 1^{1/3}e^{i5\pi/3} = 1\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1}{2}(1-\sqrt{3}i).$$



Part (b)

For a nonzero complex number $z = re^{i(\Theta+2\pi k)}$, its sixth roots are

$$z^{1/6} = \left[re^{i(\Theta+2\pi k)}\right]^{1/6} = r^{1/6}\exp\left(i\frac{\Theta+2\pi k}{6}\right), \quad k = 0, 1, 2, 3, 4, 5.$$

The magnitude and principal argument of 8 are respectively $r = 8$ and $\Theta = 0$.

$$8^{1/6} = 8^{1/6}\exp\left(\frac{i\pi k}{3}\right), \quad k = 0, 1, 2, 3, 4, 5$$

The first, or principal, root ($k = 0$) is

$$8^{1/6} = 8^{1/6}e^0 = \sqrt{2},$$

the second root ($k = 1$) is

$$8^{1/6} = 8^{1/6}e^{i\pi/3} = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}}(1+\sqrt{3}i),$$

the third root ($k = 2$) is

$$8^{1/6} = 8^{1/6}e^{i2\pi/3} = \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \sqrt{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}}(-1+\sqrt{3}i),$$

the fourth root ($k = 3$) is

$$8^{1/6} = 8^{1/6}e^{i\pi} = \sqrt{2}\,(\cos\pi + i\sin\pi) = \sqrt{2}(-1+i0) = -\sqrt{2},$$

the fifth root ($k = 4$) is

$$8^{1/6} = 8^{1/6}e^{i4\pi/3} = \sqrt{2}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \sqrt{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{1}{\sqrt{2}}(1+\sqrt{3}i),$$

and the sixth root ($k = 5$) is

$$8^{1/6} = 8^{1/6}e^{i5\pi/3} = \sqrt{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}}(1-\sqrt{3}i).$$